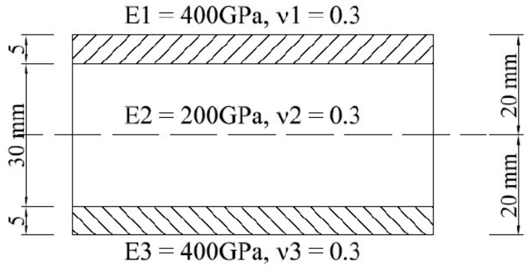
Problem T1



Datas given

The analysis focuses on a square plate, which is simply supported along its entire perimeter. The geometry and boundary conditions are defined as follows:

A Square plate with side length $a = 2$ m with Boundary Conditions simply supported on all four edges.

The plate is subjected to a uniformly distributed load (constant pressure) applied normal to its top surface: $p = 0{,}5$ kPa $= 0{,}5$ kN/m^2.

The plate is designed as a three layer structure with a total thickness $h = 40$ mm $= 0{,}04$ m. All layers are assumed to be isotropic. The mechanical properties of each layer are defined as follows:

Layer 1 (Top Face):

- Thickness: $h_1 = 5$ mm $= 0{,}005$ m
- Young's Modulus: $E_1 = 400$ GPa
- Poisson's Ratio: $\nu_1 = 0{,}3$

Layer 2 (Core):

- Thickness: $h_2 = 30$ mm $= 0{,}03$ m
- Young's Modulus: $E_2 = 200$ GPa
- Poisson's Ratio: $\nu_2 = 0{,}3$

Layer 3 (Bottom Face):

- Thickness: $h_3 = 5$ mm $= 0{,}005$ m
- Young's Modulus: $E_3 = 400$ GPa
- Poisson's Ratio: $\nu_3 = 0{,}3$

The coordinate system is centered on the mid-plane of the plate's cross section. The z-axis spans the thickness of the plate within the interval: $z \in [-h/2, h/2]$.

This convention aligns with the generalized plate model, where the displacement field and the stress resultants are defined relative to the geometric mid-surface.

To ensure numerical consistency throughout the analysis, the following units are adopted:

- Stress and Elastic Moduli: Expressed in kPa (kN/m^2).

- Bending Stiffness (D): Consequently calculated in $kN \cdot m$.
- Length and Thickness: Defined in m.

Kirchhoff Plate Model and Generalized Quantities

The structural behavior is analyzed using the Kirchhoff-Love Plate Theory. This model assumes that straight lines normal to the mid-surface remain straight, inextensible, and normal to the mid-surface after deformation.

In-plane Stress and Strain Vectors

Under the plane stress assumption for each isotropic layer, the state of stress and strain at any point is represented by the following vectors:

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad and \quad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Generalized Stress Resultants and Generalized Strains

The internal state of the plate is described by the generalized stress resultants, which are obtained by integrating the in-plane stresses $\underset{\sim}{\sigma}$ over the total thickness h.

$$\underset{\sim}{Q} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} generalized\ resultants \quad and \quad \underset{\sim}{q} = \begin{bmatrix} \eta_x \\ \eta_y \\ \eta_{xy} \\ \chi_x \\ \chi_y \\ \chi_{xy} \end{bmatrix} generalized\ strains$$

Kinematic Relationship

The relationship between the generalized strains and the generalized displacements is defined through the kinematic operator matrix

$$\underset{\sim}{\varepsilon} = \underset{\sim}{b}\, \underset{\sim}{q}, \qquad knowing\ that \quad \underset{\sim}{b} = \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 \\ 0 & 1 & 0 & 0 & z & 0 \\ 0 & 0 & 1 & 0 & 0 & z \end{bmatrix}$$

That can also be seen such as: $\underset{\sim}{b} = [\underset{\sim}{b}_0 \underset{\sim}{b}_1]$

$$\underset{\sim}{b}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underset{\sim}{I}_3 \quad and \quad \underset{\sim}{b}_1 = \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} = z\, \underset{\sim}{I}_3 \quad \text{and this is: } \underset{\sim}{b} = \begin{bmatrix} \underset{\sim}{I}_3 & z\underset{\sim}{I}_3 \end{bmatrix} \quad \underset{\sim}{b}^T = \begin{bmatrix} \underset{\sim}{I}_3 \\ z\underset{\sim}{I}_3 \end{bmatrix}$$

$$\underset{\sim}{b}^T \underset{\sim}{d}\, \underset{\sim}{b} = \begin{bmatrix} \underset{\sim}{I}_3 \\ z\underset{\sim}{I}_3 \end{bmatrix} \underset{\sim}{d}(z) \begin{bmatrix} \underset{\sim}{I}_3 & z\underset{\sim}{I}_3 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{d}(z) & z\, \underset{\sim}{d}(z) \\ z\, \underset{\sim}{d}(z) & z^2\, \underset{\sim}{d}(z) \end{bmatrix}$$

$$\underline{\underline{D}} = \int_{-h/2}^{h/2} \begin{bmatrix} \underset{\sim}{d}(z) & z\, \underset{\sim}{d}(z) \\ z\, \underset{\sim}{d}(z) & z^2\, \underset{\sim}{d}(z) \end{bmatrix} dz = \begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{B} & \mathrm{C} \end{bmatrix}$$

For definition:

$$\mathrm{A} = \int_{-h/2}^{h/2} \underset{=}{d}\,(z)\,dz$$
$$\mathrm{B} = \int_{-h/2}^{h/2} z\ \underset{=}{d}(z)\,dz = 0 \ \text{ for symmetry}$$
$$\mathrm{C} = \int_{-h/2}^{h/2} z^2\ \underset{=}{d}(z)\,dz$$

$$\boxed{\underset{=}{d}(z) = \alpha(z)\,\underset{=}{C_0}} \qquad \textit{knowing that} \quad \alpha(z) = \frac{E(z)}{1-\nu^2} \quad \textit{and} \quad \underset{=}{C_0} = \begin{bmatrix} 1 & \nu(z) & 0 \\ \nu(z) & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu(z)}{2} \end{bmatrix}$$

In this specific case, the Poisson's ratio is uniform across all layers $\nu_1 = \nu_2 = \nu_3 = 0{,}3 \Rightarrow \nu(z) = \nu = \text{cost}$, Conversely, the Young's Modulus $E(z)$ s defined as a piecewise constant function, values E_1, E_2, E_3 within their respective layers.

$$\text{obtaining:} \qquad \underset{=}{d}(z) = \frac{E(z)}{1-\nu(z)^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

$$\mathrm{A} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{=}{d}\,(z)\,dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha(z)\,\underset{=}{C_0}\,dz = \underset{=}{C_0}\int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha(z)\,dz = \underset{=}{C_0}\frac{1}{1-\nu^2}\int_{-h/2}^{h/2} E(z)\,dz$$

Given the total thickness $h = 2h_1 + h_2$ the integration domain for $E(z)$ is divided according to the coordinates of the layers relative to the mid-plane:

- Layer 1 (Top Face): $z \in \left[\frac{h_2}{2};\ \frac{h_2}{2} + h_1\right], \quad E = E_1$
- Layer 2 (Core): $z \in \left[-\frac{h_2}{2};\ \frac{h_2}{2}\right], \quad E = E_2$
- Layer 3 (Bottom Face): $z \in \left[-\frac{h_2}{2} - h_1;\ -\frac{h_2}{2}\right], \quad E = E_1$

Obtaining:

$$\int_{-h/2}^{h/2} E(z)\,dz = E_1h_1 + E_2h_2 + E_1h_1 = 2E_1h_1 + E_2h_2$$

So I define:

$$\boxed{a_1 = 2E_1h_1 + E_2h_2}$$

Obtaining at the end:

$$\mathrm{A} = \underset{=}{C_0}\frac{a_1}{1-\nu^2}$$

Exploiting the relation that takes into account C I obtain:

$$A = \frac{a_1}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} \frac{a_1}{1-\nu^2} & \frac{a_1\nu}{1-\nu^2} & 0 \\ \frac{a_1\nu}{1-\nu^2} & \frac{a_1}{1-\nu^2} & 0 \\ 0 & 0 & \frac{a_1}{2(1+\nu)} \end{bmatrix}$$

Bending Stiffness Block C

The bending block, C represents the plate's resistance to curvature. Given that the Poisson's ratio and the Young's Modulus, the components of the bending stiffness matrix are calculated as follows:

$$C = \int_{-h/2}^{h/2} z^2\, \underset{-}{d}(z)\, dz = \underset{-}{C_0} \int_{-h/2}^{h/2} z^2\, \alpha(z)\, dz = \underset{-}{C_0} \frac{1}{1-\nu^2} \int_{-h/2}^{h/2} z^2\, E(z)\, dz$$

Exploiting the relation that considers C I obtain:

$$C = \frac{b_1}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} \frac{b_1}{12(1-\nu^2)} & \frac{b_1\nu}{12(1-\nu^2)} & 0 \\ \frac{b_1\nu}{12(1-\nu^2)} & \frac{b_1}{12(1-\nu^2)} & 0 \\ 0 & 0 & \frac{b_1}{24(1+\nu)} \end{bmatrix}$$

Results for the Bending Stiffness Matrix

For a symmetric geometry, the membrane bending coupling terms vanish identically. This leads to a block-diagonal stiffness matrix, where the membrane and bending behaviors are fully uncoupled. The result for the bending stiffness matrix $\underline{\underline{D}}$ is:

$$\underline{\underline{D}} = \begin{bmatrix} \frac{a_1}{1-\nu^2} & \frac{a_1\nu}{1-\nu^2} & 0 & 0 & 0 & 0 \\ \frac{a_1\nu}{1-\nu^2} & \frac{a_1}{1-\nu^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{a_1}{2(1+\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{b_1}{12(1-\nu^2)} & \frac{b_1\nu}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & \frac{b_1\nu}{12(1-\nu^2)} & \frac{b_1}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{b_1}{24(1+\nu)} \end{bmatrix}$$

where

$$\boxed{a_1 = 2E_1h_1 + E_2h_2}$$

$$\boxed{b_1 = E_2h_2^3 + 2E_1h_1(4h_1^2 + 6h_1h_2 + 3h_2^2)}$$

Equivalent Membrane Stiffness

$$a_1 = 2E_1h_1 + E_2h_2 = 2 \cdot (400 \cdot 10^6)(0{,}005) + (200 \cdot 10^6)(0{,}03) = 10 \cdot 10^6$$

$\boxed{a_1 = 1.0 \cdot 10^7 \ \text{kN/m}}$

Equivalent Bending Parameter

$$b_1 = E_2 h_2^3 + 2E_1 h_1 (4h_1^2 + 6h_1 h_2 + 3h_2^2)$$

$E_2 h_2^3 = (200 \cdot 10^6) \cdot (0{,}03)^3 = 5{,}4 \cdot 10^3$

$4h_1^2 = 4 \cdot (0{,}005)^2 = 4 \cdot (2{,}5 \cdot 10^{-5}) = 1{,}0 \cdot 10^{-4}$

$6h_1 h_2 = 6 \cdot (0{,}005) \cdot (0{,}03) = 6 \cdot (1{,}5 \cdot 10^{-4}) = 9{,}0 \cdot 10^{-4}$

$3h_2^2 = 3 \cdot (0{,}03)^2 = 3 \cdot (9 \cdot 10^{-4}) = 2{,}7 \cdot 10^{-3}$

$\Rightarrow (4h_1^2 + 6h_1 h_2 + 3h_2^2) = 1{,}0 \cdot 10^{-4} + 9{,}0 \cdot 10^{-4} + 2{,}7 \cdot 10^{-3} = 3{,}7 \cdot 10^{-3}$

$2E_1 h_1 (\cdots) = 2 \cdot (400 \cdot 10^6) \cdot (0{,}005) \cdot (3{,}7 \cdot 10^{-3})$

$= (4 \cdot 10^6) \cdot (3{,}7 \cdot 10^{-3}) = 14{,}8 \cdot 10^3$

$\Rightarrow b_1 = 5{,}4 \cdot 10^3 + 14{,}8 \cdot 10^3 = 20{,}2 \cdot 10^3$

$\boxed{b_1 = 2{,}02 \cdot 10^4 \ \mathrm{kN \cdot m}}$

Bending Stiffness

$$\underline{\underline{D}}_{(M)} = \begin{bmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} D \end{bmatrix} \ with\ scalar\ stiffness\ D = \frac{b_1}{12(1-\nu^2)}$$

$$D_0 = \frac{b_1}{12 \cdot (1-\nu^2)}$$

$$D_0 = \frac{2{,}02 \cdot 10^4}{12 \cdot (1 - 0{,}3^2)} = \frac{2{,}02 \cdot 10^4}{12 \cdot 0{,}91} = \frac{2{,}02 \cdot 10^4}{10{,}92}$$

$$\boxed{D_0 = 1{,}8498 \cdot 10^3 \ \text{kN} \cdot \text{m}}$$

Moment-Curvature Relationships Kirchhoff

In accordance with Kirchhoff plate theory, the mechanical behavior is governed by the relationship between internal moments and the second derivatives of the transverse displacement $w(x, y)$.

The generalized curvatures χ are defined as

$$\chi_x = -\frac{\partial^2 w}{\partial x^2} \qquad \chi_y = -\frac{\partial^2 w}{\partial y^2} \qquad \chi_{xy} = -2\frac{\partial^2 w}{\partial x\, \partial y}$$

For an isotropic Layer plate where the Poisson's ratio is constant through the thickness, the bending moments M_x and M_y are related to the curvatures by the bending stiffness D_0

$$M_x = D_0\left(\chi_x + \nu\chi_y\right) \quad and \quad M_y = D_0(\chi_y + \nu\chi_x)$$

Regarding the twisting moment M_{xy}, it is expressed as:

$$M_{xy} = B\,\chi_{xy}, \qquad knowing\ that \qquad B = \frac{b_1}{24(1+\nu)}\ \text{torsional stiffness}$$

The link between the bending stiffness D_0 and the torsional stiffness B is established as follows:

$$\frac{2B}{1-\nu} = D_0 \Rightarrow B = \frac{1-\nu}{2}D_0 \Rightarrow \boxed{M_{xy} = D_0\frac{1-\nu}{2}\chi_{xy}}$$

By substituting this relationship into the expression for the twisting moment, taking into account $\chi_{xy} = -2w_{xy}$, we obtain:

$$\boxed{M_{xy} = -D_0(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y}}$$

Equilibrium Equation

To derive the governing equation of the plate, we consider the equilibrium of a differential element subjected to the transverse pressure p. The moment equilibrium equation is given by:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p$$

By substituting the expressions for the bending moments M_x, M_y and the twisting moment M_{xy}as functions of the transverse displacement $w(x,y)$, we obtain the fundamental Partial Differential Equation for the Kirchhoff plate:

$$\boxed{\nabla^4 w = \frac{p}{D_0}}$$

where:

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}$$

This equation governs the deflection of the Layer plate, where D_0 represents the equivalent flexural rigidity calculated in the previous sections.

Boundary Conditions

For a plate that is simply supported along its entire perimeter, the kinematic and static boundary conditions are defined as follows:

Displacement: $w = 0 \;\; on\ x = (0,a)\ and\ y = (0,a)$ along all edges.

Bending Moments: $M_x = 0 \;\; on\ x = (0,a) \;\; and \quad M_y = 0 \;\; on \quad y = (0,a)$

Navier's Approach

To satisfy these boundary conditions automatically, we adopt Navier's solution, expressing the transverse displacement $w(x, y)$ as a double trigonometric series**:**

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\ \sin\left(\frac{m\pi x}{a}\right)\ \sin\left(\frac{n\pi y}{a}\right)$$

This choice ensures that $w = 0$ on the boundaries. At the edges $x = (0, a)$ or $y = (0, a)$, the sin functions vanish, consequently nullifying M_x and M_y as required by the physical constraints.

Fourier Expansion of the Uniform Load

Similarly, the uniform pressure $p(x, y) = p$ is expanded into a double sine series to be compatible with the displacement field:

$$p(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P_{mn}\ \sin\left(\frac{m\pi x}{a}\right)\ \sin\left(\frac{n\pi y}{a}\right)$$

The Fourier coefficients P_{mn} are calculated using the orthogonality properties of the sine functions obtaining:

$$P_{mn} = \frac{4}{a^2}\int_0^a\int_0^a p\ \sin\left(\frac{m\pi x}{a}\right)\ \sin\left(\frac{n\pi y}{a}\right)\ dy\, dx$$

For a uniform load $p =$ cost the integral results in:

$$P_{mn} = \frac{4p}{a^2}\left[\int_0^a \sin\left(\frac{m\pi x}{a}\right)\ dx\right]\left[\int_0^a \sin\left(\frac{n\pi y}{a}\right)\ dy\right]$$

The generic integral for both dimensions is calculated as:

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\ dx = \left[-\frac{a}{m\pi}\cos\left(\frac{m\pi x}{a}\right)\right]_0^a = -\frac{a}{m\pi}(\cos(m\pi) - 1)$$

By substituting $\cos(m\pi) = (-1)^m$, the result depends on whether the index m is even or odd:

If m is even: $\Rightarrow 1 - (-1)^m = 0 \Rightarrow$ (the coefficient vanishes)

If m is odd: $\Rightarrow 1 - (-1)^m = 2 \Rightarrow \int\ = \frac{2a}{m\pi}$

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\ dx = \frac{a}{m\pi}(1 - \cos(m\pi)) = \frac{a}{m\pi}(1 - (-1)^m)$$

By combining the results for both x and y directions, the non zero Fourier coefficients for the uniform load are obtained only when both m and n are odd numbers.

$$P_{mn} = \frac{4p}{a^2}\left(\frac{2a}{m\pi}\right)\left(\frac{2a}{n\pi}\right) = \boxed{\frac{16p}{mn\pi^2}} \qquad with \quad (m, n \text{ odd})$$

Determination of the Displacement Coefficients W_{mn}

By substituting the double sine series for the displacement $w(x,y)$ and the pressure $p(x,y)$ into the governing equation $\nabla^4 w = p/D_0$, we obtain the relationship for the generalized coefficients W_{mn}

$$W_{mn} = \frac{P_{mn}}{D_0\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2\right]^2} = \frac{P_{mn}}{D_0\left(\frac{\pi^2}{a^2}(m^2+n^2)\right)^2}$$

$$\boxed{W_{mn} = \frac{P_{mn}\,a^4}{D_0\pi^4(m^2+n^2)^2}}$$

Combining the results, the full analytical expression for the displacement field of the simply supported Layer plate is

$$\boxed{w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_{mn}\,a^4}{D_0\pi^4(m^2+n^2)^2}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}$$

where $P_{mn} \neq 0$ only for odd values m, n

Moment Curvature Relationships

Using the Kirchhoff-Love plate theory, we define the relationship between the internal moments and the deformation field. The generalized curvatures are defined as

$$\chi x = -\frac{\partial^2 w}{\partial x^2} = \sum_{m,n} W_{mn}\left(\frac{m\pi}{a}\right)^2 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$
$$\chi y = -\frac{\partial^2 w}{\partial y^2} = \sum_{m,n} W_{mn}\left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$
$$\chi xy = -2\frac{\partial^2 w}{\partial x\,\partial y} = -2\cdot\sum_{m,n} W_{mn}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)$$

As a matter of compactness, I write this terms in this way:

$$k_m = m\cdot\frac{\pi}{a} \quad and \quad k_n = n\cdot\frac{\pi}{a}$$

Obtaining:

$$\chi_{x(x,y)} = \sum_{m,n} W_{mn}\cdot k_m^2\cdot\sin(k_m\cdot x)\cdot\sin(k_n\cdot y)$$

$$\chi_{y(x,y)} = \sum_{m,n} W_{mn}\cdot k_n^2\cdot\sin(k_m\cdot x)\cdot\sin(k_n\cdot y)$$

$$\chi_{xy(x,y)} = -2\cdot\sum_{m,n} W_{mn}\cdot k_m\cdot k_n\cdot\cos(k_m\cdot x)\cdot\cos(k_n\cdot y)$$

For an isotropic Layer plate with a constant Poisson's ratio, the bending and twisting moments are expressed as:

$$M_x = D \cdot (\chi_x + \nu \chi_y)$$

$$M_y = D \cdot (\chi_y + \nu \chi_x)$$

$$M_{xy} = D \cdot \frac{1-\nu}{2} \cdot \chi_{xy}$$

And this lead to:

$$M_x = -D_0 \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_{x(x,y)} = D_0 \cdot \sum_{m,n} W_{mn} \cdot (k_m^2 + \nu \cdot k_n^2) \cdot \sin(k_m \cdot x) \cdot \sin(k_n \cdot y)$$

$$M_y = -D_0 \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{y(x,y)} = D_0 \cdot \sum_{m,n} W_{mn} \cdot (k_n^2 + \nu \cdot k_m^2) \cdot \sin(k_m \cdot x) \cdot \sin(k_n \cdot y)$$

$$M_{xy} = -D_0 (1-\nu) \frac{\partial^2 w}{\partial x \, \partial y}$$

$$M_{xy(x,y)} = -D_0 \cdot (1-\nu) \cdot \sum_{m,n} W_{mn} \cdot k_m \cdot k_n \cdot \cos(k_m \cdot x) \cdot \cos(k_n \cdot y)$$

Shear Forces T_x, T_y

The transverse shear forces are derived from the equilibrium of the plate element by differentiating the moment fields:

$$\boxed{T_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \qquad and \qquad T_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}}$$

$$\frac{\partial M_x}{\partial x} = D \cdot \sum_{m,n} W_{mn} \cdot (k_m^2 + \nu \cdot k_n^2) \cdot k_m \cdot \cos(k_m \cdot x) \cdot \sin(k_n \cdot y)$$

$$\frac{\partial M_{xy}}{\partial y} = D \cdot (1-\nu) \cdot \sum_{m,n} W_{mn} \cdot k_m \cdot k_n^2 \cdot \cos(k_m \cdot x) \cdot \sin(k_n \cdot y)$$

$$T_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = D \cdot \sum_{m,n} W_{mn} \cdot k_m \cdot \cos(k_m \cdot x) \cdot \sin(k_n \cdot y) \cdot [(k_m^2 + \nu \cdot k_n^2) + (1-\nu) \cdot k_n^2]$$

If i consider:

$$[(k_m^2 + \nu \cdot k_n^2) + (1-\nu) \cdot k_n^2] = k_m^2 + k_n^2$$

Then:

$$\Rightarrow T_{x(x,y)} = D \cdot \sum_{m,n} W_{mn} \cdot k_m \cdot (k_m^2 + k_n^2) \cdot \cos(k_m \cdot x) \cdot \sin(k_n \cdot y)$$

Then omitting the passages, considering the symmetry of the element:

$$T_{y(x,y)} = D \cdot \sum_{m,n} W_{mn} \cdot k_n \cdot (k_m^2 + k_n^2) \cdot \sin(k_m \cdot x) \cdot \cos(k_n \cdot y)$$

The global equilibrium is satisfied when the sum of the shear force gradients matches the external pressure

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + p = 0.$$

Edge Reactions

In thin plate theory, the twisting moment M_{xy} at the boundary cannot be satisfied as an independent boundary condition. Therefore, it is mathematically combined with the shear force to define the effective edge reactions, known as Kirchhoff shears T_{xx} *and* T_{yy}:

$$\boxed{T_{xx} = T_x + \frac{\partial M_{xy}}{\partial y}} \text{(Reaction on edges with normal } x\text{)}$$

$$\boxed{T_{yy} = T_y + \frac{\partial M_{xy}}{\partial x}} \text{(Reaction on edges with normal } y\text{)}$$

From the upper part we have gained the values for $\frac{\partial M_{xy}}{\partial y}$ and $\frac{\partial M_{xy}}{\partial x}$ in terms of the series, so we can compute the value for T_{xx} and T_{yy} such as:

$$T_{xx(x,y)} = D \cdot \sum_{m,n} W_{mn} \cdot k_m \cdot \cos(k_m \cdot x) \cdot \sin(k_n \cdot y) \cdot [(k_m^2 + \nu \cdot k_n^2) + (1-\nu) \cdot k_n^2]$$

$$T_{yy(x,y)} = D \cdot \sum_{m,n} W_{mn} \cdot k_n \cdot \sin(k_m \cdot x) \cdot \cos(k_n \cdot y) \cdot [(k_n^2 + \nu \cdot k_m^2) + (1-\nu) \cdot k_m^2]$$

And simplifying the computations, we obtain:

$$T_{xx(x,y)} = D \cdot \sum_{m,n} W_{mn} \cdot k_m \cdot (k_m^2 + k_n^2) \cdot \cos(k_m \cdot x) \cdot \sin(k_n \cdot y)$$
$$T_{yy(x,y)} = D \cdot \sum_{m,n} W_{mn} \cdot k_n \cdot (k_m^2 + k_n^2) \cdot \sin(k_m \cdot x) \cdot \cos(k_n \cdot y)$$

These resultants represent the actual force transmitted from the plate to the supports along the boundary.

For this specific rectangular geometry, the distributed reactions along the supports are:

- On edges $x = 0$ and $x = a$ (normal to x)

$$RX_0(y) = T_{nn}\,|_{x=0} = T_x(0,y) + \frac{\partial M_{xy}}{\partial y}\,|_{x=0}$$
$$RX_a(y) = T_{nn}\,|_{x=a} = T_x(a,y) + \frac{\partial M_{xy}}{\partial y}\,|_{x=a}$$

- On edges $y = 0$ and $y = a$ (normal to y)

$$RY_0(x) = T_{nn}\,|_{y=0} = T_y(x,0) + \frac{\partial M_{xy}}{\partial x}\,|_{y=0}$$

$$RY_a(x) = T_{nn}\ |_{y=a} = T_y(x,a) + \frac{\partial M_{xy}}{\partial x}\ |_{y=a}$$

As we have computed in the upper part.

Maximum and Comparison with a Homogeneous Plate

the performance of the Layer plate is evaluated by comparing it against a reference homogeneous plate composed entirely of Material 2 (the core material), with a total thickness of $h = 40\ mm =\ $0,04m)

For a homogeneous isotropic plate, the flexural rigidity D_1 is calculated based on its uniform Young's Modulus E_2 and constant thickness h

$$\boxed{D_1 = \frac{E_2 h^3}{12(1-\nu^2)}}$$

Given the following numerical parameters for Material 2:

- Thickness: $h_2 = 40$ mm $= 0{,}04$ m
- Young's Modulus: $E_2 = 200$ GPa
- Poisson's Ratio: $\nu_2 = 0{,}3$

$$D_1 = \frac{(200 \cdot 10^6) \cdot (0{,}04^3)}{12 \cdot (1 - 0{,}3^2)} = \frac{(200 \cdot 10^6) \cdot (64 \cdot 10^{-6})}{10{,}92} = \frac{12{,}8 \cdot 10^3}{10{,}92}$$

$$\boxed{D_1 = 1{,}17216 \cdot 10^3\ \text{kN} \cdot \text{m}}$$

Maximum Displacement $w_{\max}$

For a square, simply supported plate under a uniform load the maximum deflection occurs at the center $(a/2, a/2)$. Utilizing the Navier solution series, the maximum displacement $w_{\max}$ is expressed as:

$$w_{\max} = \frac{16pa^4}{D\pi^6} \sum_{\substack{m,n=1 \\ m,n \text{ dispari}}}^{\infty} \frac{1}{mn(m^2+n^2)^2}$$

The numerical evaluation of the double series converges to

$$\sum_{m,n \text{ odd}} \frac{1}{mn(m^2+n^2)^2} = 0{,}27975$$

Consequently, the maximum displacement can be simplified using a dimensionless coefficient

$$\Rightarrow w_{\max} = \alpha \frac{pa^4}{D}$$

With

$$\alpha = \frac{16}{\pi^6} \cdot \sum_{\substack{m,n=1 \\ m,n \text{ odd}}}^{\infty} \frac{1}{mn(m^2+n^2)^2} = 0{,}0043$$

Numerical Results

To finalize the analysis, a direct comparison is performed between the Layer plate and the reference homogeneous plate to determine the structural efficiency of the layered configuration.

The displacement is calculated using the following parameters:

Uniform Pressure $p = 0{,}5$ kN/m^2

Plate Dimension $a = 2\ m$

Load Term $pa^4 = 0{,}5 \cdot 16 = 8$ kN/m^2

Layer Configuration (D = D_0)

Using the calculated flexural rigidity for the Layer plate D_0 = 1,8498 · 10^3

$$w_{\max} = \alpha \frac{8}{D_0} = 0{,}0043 \cdot \frac{8}{1{,}8498 \cdot 10^3} = 1{,}859 \cdot 10^{-5}\ \text{m}$$
$$\boxed{w_{\max} = 1{,}86 \cdot 10^{-5}\ \text{m} = 0{,}0186\ \text{mm}}$$

Homogeneous Reference (D = D_1)

Using the flexural rigidity of the homogeneous plate composed of Material 2 D_1 = 1,1722 · 10^3:

$$w_{\max} = \alpha \frac{8}{D_1} = 0{,}0043 \cdot \frac{8}{1{,}1722 \cdot 10^3} = 2{,}9346 \cdot 10^{-5}\ \text{m}$$
$$\boxed{w_{\max} = 2{,}93 \cdot 10^{-5}\ \text{m} = 0{,}0293\ \text{mm}}$$

Conclusion

The efficiency of the Layer design is quantified by the ratio of the maximum displacements:

$$\boxed{\frac{w_{\max}}{w_{\max}} = \frac{D_1}{D_0} = 0{,}6337}$$

This result demonstrates that the Layer plate is significantly stiffer than a solid plate of the same total thickness made entirely of Material 2. Specifically, the Layer configuration achieves a 36.6% reduction in maximum deflection compared to the homogeneous reference

Maximum Bending Moment Analysis $M_{\max}$

The moments result as similar because while $w \propto 1/D$, the moment $M \propto D$, causing D to cancel out in the final moment expression.

At the plate center $(a/2, a/2)$, the moment is:

$$M_{x\left(\frac{a}{2},\frac{a}{2}\right)} = \left(16 \cdot p \cdot \frac{a^2}{\pi^4}\right) \cdot \sum_{m,n=1}^{\infty} \left(\frac{m^2 + \nu \cdot n^2}{m \cdot n \cdot (m^2 + n^2)^2}\right) \cdot \sin\left(m \cdot \frac{\pi}{2}\right) \cdot \sin\left(n \cdot \frac{\pi}{2}\right)$$

considering that for odd indices (m, n odd), the trigonometric terms become:

$$\sin\left(m \cdot \frac{\pi}{2}\right) = \pm 1, \qquad \sin\left(n \cdot \frac{\pi}{2}\right) = \pm 1 \quad (m, n\ dispari)$$

The analytical expression for the moment M_x at this location is given by the following double sine series

$$\left|M_x\left(\frac{a}{2},\frac{a}{2}\right)\right| = \frac{16pa^2}{\pi^4}\sum_{m,n\text{ dispari}} \frac{m^2+\nu n^2}{mn(m^2+n^2)^2}$$

To facilitate numerical calculation, the absolute value of the maximum moment can be expressed using a dimensionless coefficient β

$$\boxed{M_{x,\max} = \beta\, pa^2}$$

That is for definition:

$$\boxed{\beta(\nu) = \frac{16}{\pi^4}\sum_{m,n\text{ dispari}}^{\infty} \frac{m^2+\nu n^2}{mn(m^2+n^2)^2}}$$

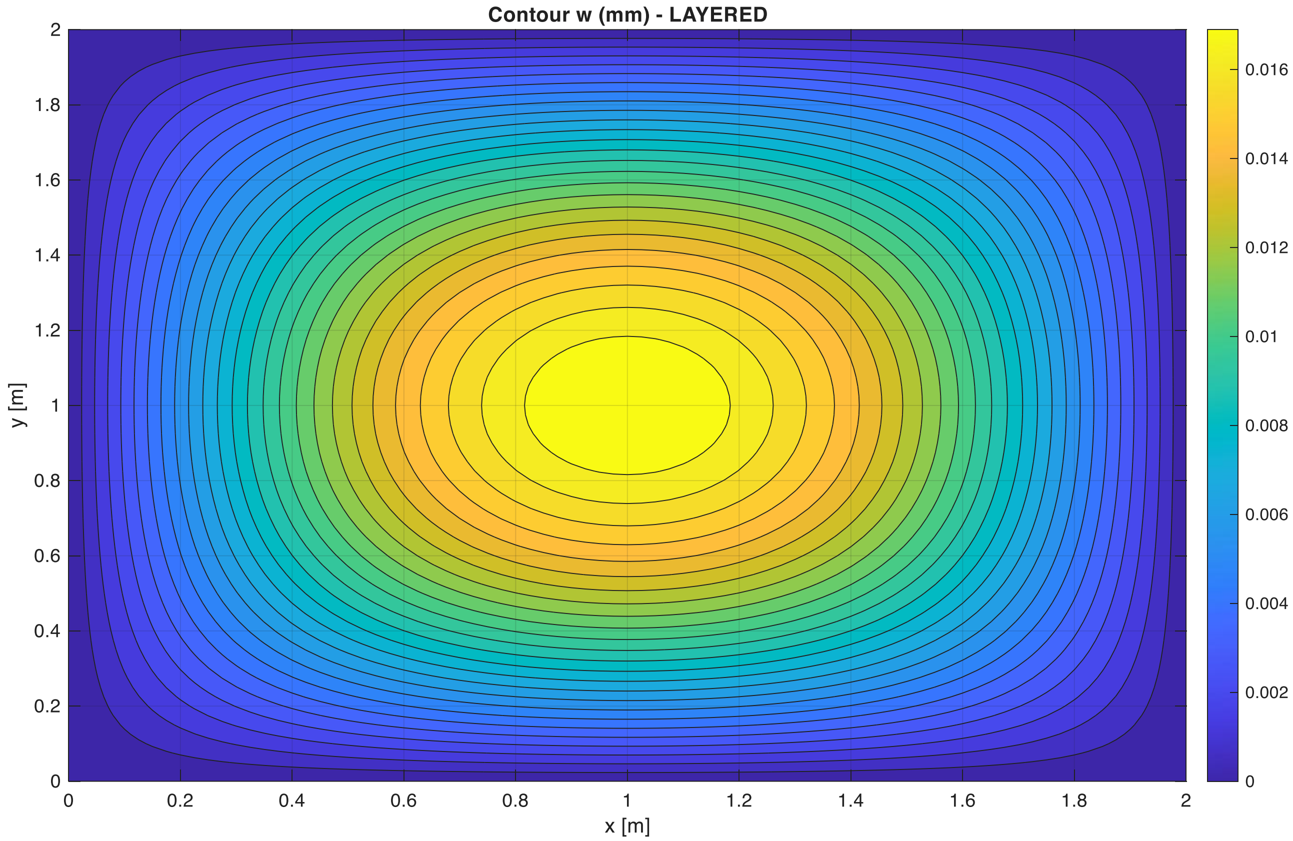
For $\nu = 0{,}3$, the numerical coefficient is

$$\beta = 0{,}065638 \Rightarrow M_{x,max} = \beta\, pa^2$$

With $pa^2 = 0{,}5 \cdot 4 = 2$ kN, the maximum moments are:

$$\boxed{M_{x,\max} = M_{y,\max} = 0{,}065638 \cdot 2 = 0{,}1313 \text{ kN}}$$

Since ν is identical in both the Layer and homogeneous cases, the maximum bending moments are equal under the same load and boundary conditions.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x [m]
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0.4
0.6
0.8
1
1.2
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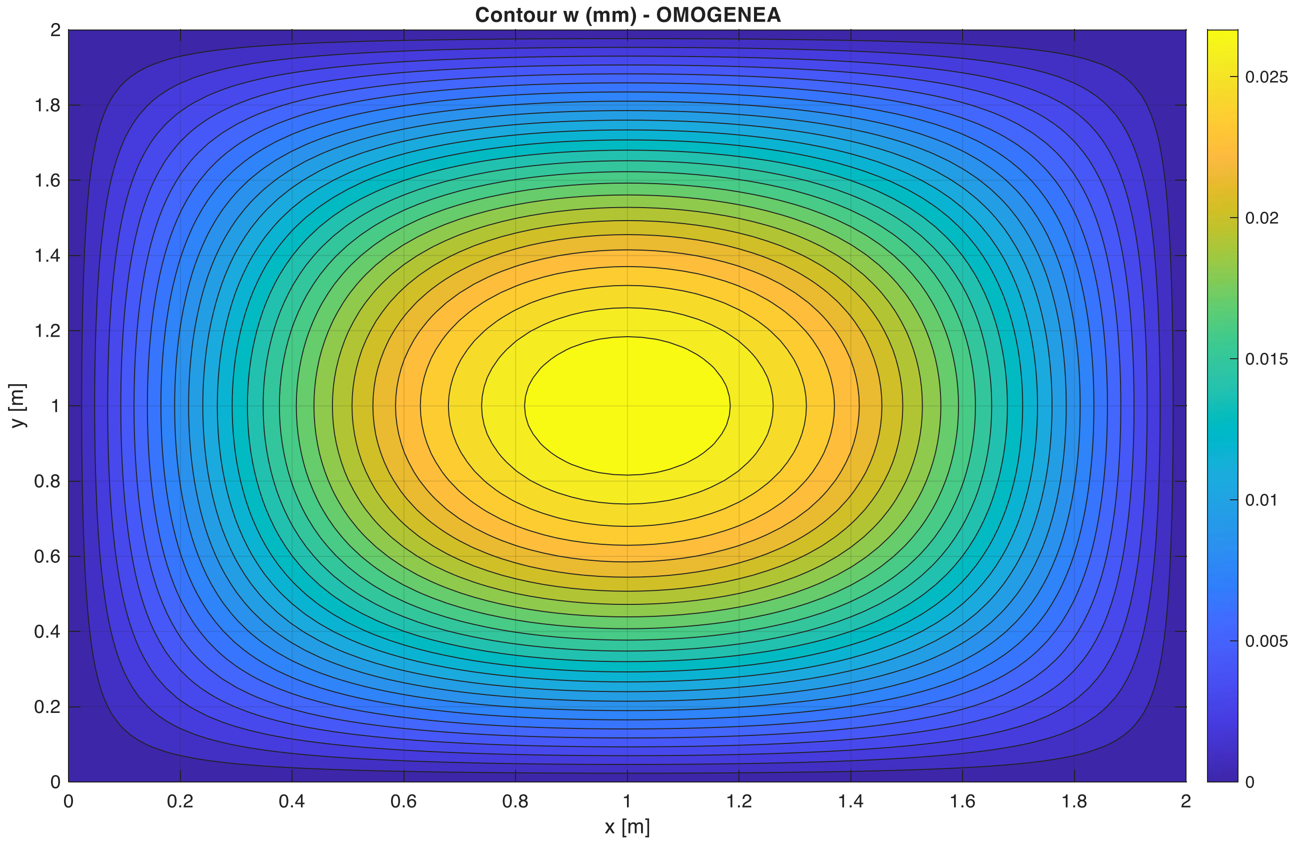
1.6
1.8
2
0.002
0.004
0.006
0.008
0.01
0.012
0.014
0.016


Contour w (mm) - OMOGENEA
y [m]
x [m]
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0.4
0.6
0.8
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1.2
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1.6
1.8
2
0.005
0.01
0.015
0.02
0.025